Gauge Independent Lagrangian Reduction of Constrained Systems

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Abstract

A gauge independent method of obtaining the reduced space of constrained dynamical systems is discussed in a purely lagrangian formalism. Implications of gauge fixing are also considered.

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An important aspect in the hamiltonian formulation of gauge theories is to obtain the reduced (physical) space comprising the true canonical variables. This is usually done by fixing a gauge that removes the unphysical degrees of freedom [1, 2]. To avoid the ambiguities and arbitrariness inherent in the gauge fixing procedure it becomes desirable to abstract the reduced space in a gauge independent manner [3]. This also helps in defining a class of admissible or allowed gauges as those which yield a reduced space that is equivalent, modulo canonical transformations, to the one obtained in the gaugeless scheme [4]. Now for a system of constraints in strong involution (as happens, for example, in abelian gauge theories) there is a definite gauge independent hamiltonian method of reducing the degrees of freedom that is based upon the Levi-Civita transformation [5]. This idea has been exploited to obtain the reduced phase space of several models [4, 6, 7].

In this paper, using certain results from the theory of differential equations, a purely lagrangian approach for obtaining the reduced space in a gauge independent manner will be discussed. Consequently it provides a lagrangian realisation of the Levi-Civita reduction process [4, 5, 6, 7]. Moreover the proposed method is more direct and does not require the Dirac [1] algorithm for computing the constraints or their classification into first and second class, which is an essential perquisite for the hamiltonian Levi-Civita method. In this sense the present analysis is similar in spirit to the symplectic approach [8] based on Darboux theorem but, contrary to it, does not need a first order lagrangian as the starting point. Indeed both first and second order systems will be discussed here on an identical footing. The effect of gauge fixing is also considered. It is shown that the hamiltonian formalism admits a wider class of allowed gauges compared to the present lagrangian formalism. This exercise illuminates, if not settles, the debate [4, 9, 10] regarding the simultaneous imposition of the axial ($A_3 \approx 0$) and temporal ($A_0 \approx 0$) gauges in pure electrodynamics.

From the theory of differential equations unsolvable with respect to the highest derivatives, it is possible to express the lagrange equations for second order systems with variables v by an equivalent set of independent equations [3],

$$\begin{aligned} \ddot{p} &= \Theta(p, \dot{p}, q, \beta, \dot{\beta}, \ddot{\beta}) \\ \dot{q} &= \Phi(p, \dot{p}, q, \beta, \dot{\beta}) \\ r &= \Psi(p, q, \beta) \end{aligned} \tag{1}$$

where $v = (p, q, r, \beta)$ and Θ, Φ, Ψ are some functions of the indicated arguments. In a nonsingular theory q, r, β are absent so that there is an unconstrained dynamics with $\ddot{p} = \Theta(p, \dot{p})$. For singular theories the last two equations of (1) represent the constraints. Now recall that the lagrange equations were derived by a variational principle on the assumption that all $v, \dot{v}$ were free. Since the constraints impose certain restrictions on $v, \dot{v}$, it is essential that these keep the set of equations (1) unmodified, or internal consistency is lost. Consequently time derivatives of the constraints must vanish by virtue of these equations. This implies that the complete constraint sector is contained in (1). It avoids the Dirac ([1]) algorithm of iteratively generating this sector in the hamiltonian formalism. Note also that the absense of any equation for β indicates a possible degeneracy in (1).

The idea is now to pass from the constrained $v = (p, q, r, \beta)$ to the unconstrained $v = p$ by removing q, r, β. The variable r can be trivially eliminated in favour of p, q, β. In the physically interesting gauge systems the constraints are implemented by a lagrange multiplier whose time derivative, therefore, does not appear in the lagrangian. This multiplier is identified with q which can thus be removed in favour of p, β by using (1). The lagrangian in the reduced sector is now a function of $(v, \dot{v}; v = p, \beta)$. By evaluating the lagrange equations in this sector it is possible to recognise β as the variable that does not occur in these equations. This suggests a specific polarisation of the reduced variables that isolates β eliminating it automatically from the lagrangian and its final unconstrained form is obtained. The physical hamiltonian is now found by performing the standard Legendre transformation.

The same analysis is now applied for first order systems. The form of the lagrange equations analogous to (1) is given by,

$$\begin{aligned} \dot{p} &= \Phi(p, \beta, \dot{\beta}) \\ q &= \Psi(p, \beta) \end{aligned} \tag{2}$$

where $v = (p, q, \beta)$ is the set of variables. Contrary to the earlier case there is only one constraint, given by the second equation in the above set. It is now straightforward to reduce the degrees of freedom by mimicing the previous steps.

An interesting feature is the crucial role played by the first order equations which, in a conventional analysis [1, 2, 3], are always taken as constraints. Here, on the contrary, if such equations occur in a second order system

(1) these are regarded as constraints whereas, in a first order system (2), these are true equations of motion. The latter interpretation is also valid even in those cases where a first order system occurs as a subsystem of a second order system as, for example, happens with the matter sector in spinor electrodynamics. Incidentally, the inappropriateness of considering any first order equation as a constraint was also observed in the symplectic viewpoint [8].

To illustrate the above ideas in a simpler setting consider a nondegenerate singular theory so that the variable β is absent. This corresponds to a second class theory in Dirac's [1] nomenclature. A typical example is provided by the Proca model,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu \tag{3}$$

The equations of motion are,

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \tag{4}$$

whose zero component is the constraint,

$$(\partial^2 - m^2)A_0 - \partial_0(\partial_i A_i) = 0 \tag{5}$$

Time derivative of this constraint vanishes by virtue of the equation of motion revealing the internal consistency of the model. It is clear that A_0 gets identified with q (1). Eliminating A_0 from (3) by using (5), the unconstrained lagrangian is obtained,

$$\mathcal{L} = -\frac{1}{4}F_{ij}^2 + \frac{1}{2}\dot{A}_i^2 + \frac{1}{2}\partial_i\dot{A}_i\frac{1}{\partial^2 - m^2}\partial_j\dot{A}_j - \frac{m^2}{2}A_i^2 \tag{6}$$

The reduced hamiltonian, derived by a standard Legendre transform from the above lagrangian, is given by,

$$H = \frac{1}{2}\int d^3x(\pi_i^2 + \frac{1}{m^2}(\partial_i\pi_i)^2 + m^2A_i^2 + \frac{1}{2}F_{ij}^2) \tag{7}$$

where (π_i, A^i) are the canonical variables. It reproduces the expression obtained from the Dirac anlysis of eliminating second class constraints by Dirac brackets and showing that these brackets reduce to the Poisson brackets for the canonical variables [3]. All these details are unnecessary in the present

context. Moreover explicit conversion of (3) to a first order form, as is required in the symplectic approach [8], is avoided.

Next consider the more interesting case of a degenerate singular theory, a classic example of which is spinor electrodynamics,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m - eA\!\!\!/)\psi \tag{8}$$

The equations of motion are,

$$\begin{aligned} (i\partial\!\!\!/ - m - eA\!\!\!/)\psi &= 0 &(9)\\ \partial^\alpha F_{\alpha\mu} - j_\mu &= 0 &(10) \end{aligned}$$

where $j_\mu = e\bar{\psi}\gamma_\mu\psi$ is the current. Although (9) is first order it is not a constraint since the matter sector is first order. Only the $\mu = 0$ component of (10) is a constraint. Furthermore there is a degeneracy in these equations which follows from current conservation. As before, the multiplier A_0 (identified with q) can be eliminated in favour of the other variables by solving the constraint. Using this, (8) is expressed in terms of the reduced set of variables. The Lagrange equations in these variables are,

$$\partial^j F_{ji} + \partial_0^2[(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2})A_j] + \frac{\partial_i}{\partial^2}\partial_0 j_0 - j_i = 0 \tag{11}$$

It is obvious that the variable β, manifesting the degeneracy, is just the longitudinal (L)- component of A_i, since it has dropped out from (11). Consequently by choosing the orthogonal polarisation $A_i = A_i^T + A_i^L$ the lagrangian gets further reduced,

$$\mathcal{L} = \frac{1}{2}\dot{A}_i^{T2} - \frac{1}{4}F_{ij}^2(A^T) + \frac{1}{2}j_0\frac{1}{\partial^2}j_0 + j_iA_i^T + \mathcal{L}_M \tag{12}$$

where, expectedly, A_i^L gets automatically removed and $\mathcal{L}_M$ is the pure matter part. Denoting the two independent components of A_i^T by $a_I(I = 1, 2)$;

$$A_i^T = (\delta_{iI} - \delta_{i3}\frac{\partial_I}{\partial_3})a_I \tag{13}$$

the lagrangian (8) is finally expressed in terms of the independent unconstrained variables (a_I, ψ). The reduced (physical) hamiltonian, obtained by

taking a Legendre transform of this lagrangian, is given by,

$$H = \int d^3x \{ \frac{1}{2}[(\delta_{iI} - \frac{\partial_i \partial_I}{\partial^2})p_I]^2 + \frac{1}{4}F_{ij}^2(a) - \frac{1}{2}j_0 \frac{1}{\partial^2} j_0 - j_I a_I + j_3 \partial_3^{-1} \partial_I a_I \} + H_M \tag{14}$$

directly in terms of the independent canonical pairs $(a^I, p_I), (\psi, \psi^*)$, and H_M is the pure matter contribution. This reduced space coincides with that obtained in the Hamiltonian formalism of abstracting the canonical set by a Levi-Civita transformation and then evaluating the total hamiltonian on the constraint surface [3].

An analogous treatment, which will also illuminate the connection with the symplectic approach [8] based on the Darboux transformation, is now given by converting (8) to a first order form,

$$\mathcal{L} = -\frac{1}{4}F_{ij}^2 - \frac{1}{2}\pi_i^2 - \pi_i F_{0i} + \bar{\psi}(i\not{\partial} - m - e\not{A})\psi \tag{15}$$

Proceeding as discussed for general first order systems (2) one solves the constraint and arrives at a reduced lagrangian,

$$\mathcal{L} = -\pi_i^T \dot{A}_i^T - \frac{1}{2}\pi_i^{T2} - \frac{1}{4}F_{ij}^2(A^T) + \frac{1}{2}j_0 \frac{1}{\partial^2} j_0 + j_i A_i^T + \mathcal{L}_M \tag{16}$$

It is now essential to abstract the *independent* canonical pairs from (π_i^T, A_i^T) to obtain the final reduced space. A possible way is to choose an arbitrary polarisation for these variables from which the reduced hamiltonian may be derived by a Lagendre transform. This circuitous path is avoided by making the Darboux transformation, which comprises (13) together with,

$$\pi_i^T = (\delta_{iI} - \frac{\partial_i \partial_I}{\partial^2})p_I \tag{17}$$

so that the canonical 1-form remains diagonalised $(i.e. \pi_i^T \dot{A}^{iT} = p_I \dot{a}^I)$ and the reduced hamiltonian directly read off from (16) reproduces (14).

Finally the issue of gauge fixing is considered with a view to reveal the subtleties in the simultaneous implementation of $A_3 = 0$ and $A_0 = 0$ in pure electrodynamics [3, 4, 9, 10]. Before that it is worthwhile to point out that the Coulomb gauge $\partial_i A_i = 0$ is the most natural choice since it implies the removal of A_i^L which was gauge independently identified as the redundant (β) variable. Effectively, therefore, choosing the Coulomb gauge is like not

choosing any gauge. Now confining our attention to the pure Maxwell theory, it is found from (10) that the solution for the multiplier consistent with the axial ($A_3 = 0$) gauge is,

$$A_0 = \frac{\partial_0}{\partial^2}(\partial_I A_I); I = 1, 2 \tag{18}$$

Eliminating both A_3 and A_0 from (8) yields the unconstrained lagrangian which, after the usual Legendre transform, leads to the reduced hamiltonian,

$$H_{axial} = \int d^3x \frac{1}{2}[(\frac{\partial_I}{\partial_3}\pi_I)^2 + \pi_I^2 - A_I\partial^2 A_I - (\partial_I A_I)^2] \tag{19}$$

with (A_I, π^I) being the canonical pairs. It is simple to check that the canonical transformation,

$$\pi_I = -\sqrt{-\partial^2}a_I; A_I = \frac{1}{\sqrt{-\partial^2}}p_I \tag{20}$$

establishes the equivalence of (19) with the gauge independent result (14). Hence the axial gauge supplemented with (18) is an "allowed" choice. If, on the contrary, $A_0 = 0$ is taken instead of (18) then the reduced hamiltonian is now given by (19), but without its first term. Hence canonical equivalence with (14) cannot be shown so that the choice $A_0 = 0$ with $A_3 = 0$ is disallowed.

The picture is different in the hamiltonian analysis where the familiar expression for the total hamiltonian is given by,

$$H_T = \int d^3x(\frac{1}{2}\pi_i^2 + \frac{1}{4}F_{ij}^2 + A_0\partial_i\pi_i + \lambda\pi_0) \tag{21}$$

If the constraints $\pi_0 = 0, \partial_i\pi_i = 0$ are implemented by fixing $A_0 = 0$ and $A_3 = 0$, then the reduced hamiltonian obtained from (21) exactly corresponds to (19). Moreover the Dirac brackets of A_I, π^I are identical to their Poisson brackets so that these variables constitute the canonical set [2, 3]. Contrary to the lagrangian formulation, therefore, the axial-temporal gauge is a valid choice in the hamiltonian framework. Moreover the axial gauge imposed together with $\partial_3 A_0 - \pi_3 = 0$, which is the hamiltonian analogue of (18), also yields a valid reduced space [2, 3]. Consequently the hamiltonian formalism admits a wider class of admissible gauges than the lagrangian formalism.

It is clear that the practical viability of this scheme depends on solving the constraint. While this can always be done in abelian theories (including Chern-Simons terms) and the redundant (β) variable identified with A_i^L, the same cannot be said for nonabelian theories involving nonlinear constraints. This difficulty, it is emphasised, is more technical than conceptual. The present method, however, suggests an intriguing possibility of solving the nonlinear constraint and identifying β by a perturbative series around the known abelian results. It should be mentioned that even in the hamiltonian formulation it is problematic to generalise the Levi-Civita [5] method to systematically reduce the degrees of freedom in a nonabelian theory without gauge fixing [4, 6]. Nevertheless, in those cases where a gauge independent reduction is constructively feasible, this approach provides a definite simplification over the elaborate hamiltonian formalism based on the Levi-Civita transformation [4, 5, 6, 7]. Furthermore, by discussing both second and first order systems within a unified framework, it dispenses with the Darboux transformation [8].

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References

[1] P.A.M. Dirac, "Lectures on Quantum Mechanics" (Belfer Graduate School of Science, Yeshiva Univ. New York, 1964.)

[2] A. Hanson, T. Regge and C. Teitelboim, "Constrained Hamiltonian Systems" (Academia Nazionale dei Lincei, Rome, 1976.)

[3] D. Gitman and I. Tyutin, "Quantisation of Fields with Constraints" (Springer- Verlag, 1990)

[4] S. Gogilidze, A. Khvedelidze and V.Pervushin, Phys. Rev. D53, 2160 (1996).

[5] T.Levi-Civita, Prace Mat.-Fiz 17, 1 (1906).

[6] S.Shanmugadhasan, J. Math. Phys. 14, 677 (1973).

[7] J.Goldberg, E.Newman and C.Rovelli, J. Math. Phys. 32, 2739 (1991); Also see D.Boyanovsky, E.Newman and C.Rovelli, Phys. Rev. D45, 1210 (1992).

[8] For a modern perspective, see R.Jackiw in "Diverse Topics in Theoretical and Mathematical Physics" (World Scientific, Singapore, 1995).

[9] R.Sugano and T.Kimura, J. Phys. A16, 4417 (1983) and references therein.

[10] M.Lavelle and D.McMullan, Phys. Lett. B316, 172 (1993).